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HYDRODYNAMICS OF A VORTEX RING

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We considered the kinematics and dynamics of a vortex ring in an incompressible fluid in toroidal coordinates. We obtained the change in the pressure difference along the boundary between two flow regions in the case of a moving torus.

Vortex rings, or vorticity, rank among the most important properties of liquid and gas that underlie diversified forms of their motion. The best example of the formation and development of vortex rings are coherent structures in turbulent flows that represent an expressed form of the concentration of vorticity rolled up into a torus [1]. Such a structure is stable, has a large lift force, and in motion over large distances preserves the momentum imparted to it [2].

Vortex rings form the hydrodynamic basis in a number of new technologies, including chemical industries [3, 4] and the cleaning of objects in machines under repair [5, 6]. The problem of the formation and development of rings was the concern of [7-15]. We note that in all of the works mentioned the solution of the problem is constructed in a cylindrical system of coordinates. The introduction of toroidal coordinates makes it possible to obtain certain new solutions along with the well-known ones.

In the present work we consider the kinematics and dynamics of a vortex ring in toroidal coordinates (σ, τ, φ) [16].

We obtained the solution of the problem concerning the development of a vortex ring in a homogeneous rectilinear fluid flow.

Kinematics of a Vortex. We assume that a particle of the fluid of a vortex ring moves along the surface of the torus of the indicated coordinate system (σ, τ, φ). Then, the streamline is characterized by the coordinate lines τ = const and φ = const, whereas the components of the velocity vector $\overline{V}(V_\sigma, V_\tau, V_\varphi)$ are respectively equal to

$$V_\sigma = V_\sigma; \quad V_\tau = V_\varphi = 0 .$$

The centerline of the torus is the axis $\tau = 0$. This makes it possible to exclude from the continuity equation the variable φ and the terms containing the derivative $\partial/\partial\varphi$. Then the mass conservation law in the case of an incompressible fluid has the form

$$\operatorname{div} \overline{V} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \sigma} \left(V_\sigma \sqrt{\left(\frac{g}{g_{\sigma\sigma}} \right)} \right) = 0 .$$

As a result of integration under the boundary conditions

$$V_\sigma \to 0 \quad \text{when} \quad \tau \to \infty ; \quad \sigma \to 0 , \quad \tau \to 0 ;$$

$$V_\sigma = V_0 = \text{const} \quad \text{when} \quad \sigma = \pm \pi , \quad \tau \to 0$$

we obtain an expression for the velocity

$$V_\sigma = \frac{V_0 \, (\operatorname{ch} \tau - \cos \sigma)^2}{4 \operatorname{ch}^3 \tau} . \tag{1}$$

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In Cartesian coordinates x, y, z at $x = 0$ we write expression (1) according to the transition formulas [16]:

$$V_\sigma = \frac{V_0 a^4 \sqrt{(y^2 + z^2 - a^2)^2 + (2az)^2}}{(y^2 + z^2 + a^2)^3}.$$

In cylindrical coordinates z, y, φ, where $y^2 = r^2 = x^2 + y^2$, according to the transition formulas $U_z = V_\sigma \cos\alpha$, $U_y = -V_\sigma \sin\alpha$, we have

$$U_z = \frac{V_0 a^4 (y^2 - z^2 - a^2)}{(y^2 + z^2 + a^2)^3}, \tag{2}$$

$$U_y = -\frac{2V_0 a^4 yz}{(y^2 + z^2 + a^2)^3}, \tag{3}$$

or for the stream function $\Psi_0(U_y = -1/y\cdot(\delta\Psi_0/\delta z);\ U_z = 1/y\cdot(\delta\Psi_0/\delta y))$:

$$\Psi_0 = \frac{-V_0 a^4 y^2}{2(y^2 + z^2 + a^2)^2}. \tag{4}$$

Along the line $y = 0$ (i.e., the 0z axis) we find the circulation of velocity along the closed contour bounding the region $\infty > y \geq 0$, $\infty > z > -\infty$ from the formula

$$\Gamma = -\int_{-\infty}^{\infty} U_z\,dz.$$

Substituting the expression for U_z from Eq. (2) at $y = 0$ we obtain the value $\Gamma = V_0 \pi a/2$. Whence we determine the value of the velocity on the axis of symmetry at the center of the torus:

$$V_0 = \frac{2\Gamma}{\pi a}. \tag{5}$$

Thus, the velocity V_0 is directly proportional to the vortex intensity and inversely proportional to its radius.

Dynamics of the Vortex Ring Developing in a Homogeneous Rectilinear Fluid Flow. We assume that the vortex ring moves in an incompressible fluid along the axis of symmetry 0z of the vortex with the velocity U_∞. The solution of this problem can be represented as a superposition of two potential flows: flow around a stationary annular vortex by a fluid with a constant velocity $U_\infty = V_1 = \text{const}$ and the motion of the fluid along the torus surface initiated by the vortex. According to the superposition principle [17], the fluid stream function in a moving vortex ring can be presented as follows:

$$\Psi = \Psi_0 + \Psi_1.$$

Here Ψ_0 has been determined according to formula (4). The stream function Ψ_1 in a cylindrical coordinate system

$$U_y = -\frac{1}{y}\frac{\partial\Psi_1}{\partial z}, \quad U_z = \frac{1}{y}\frac{\partial\Psi_1}{\partial y}$$

of a homogeneous rectilinear potential flow ($U_y = U_\varphi = 0$, $U_z = U_\infty = V_1 = \text{const}$) is equal to

$$\Psi_1 = V_1 \frac{y^2}{2}.$$

Finally, for the stream function Ψ we have

$$\Psi = V_1 \frac{y^2}{2} - \frac{V_0 a^4 y^2}{2(y^2 + z^2 + a^2)^2}. \tag{6}$$

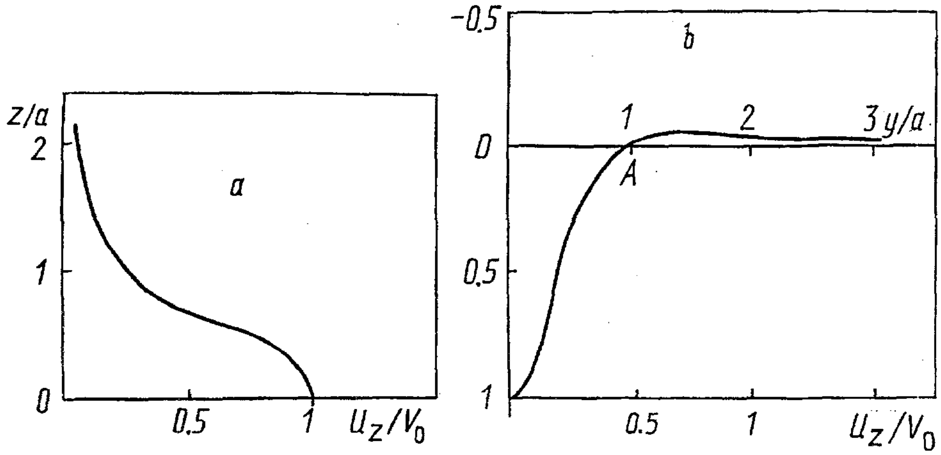

Fig. 1. Variation of velocity U_z/V_0 along the coordinate axes: a) along the z/a axis; b) along the y/a axis.

Analysis of the Results Obtained. In Fig. 1 we present the results of calculation of the dependences of the velocity component U_z along the vortex axis $0z$ (Fig. 1a) and across it, i.e., on the $0y$ axis (Fig. 1b) obtained from formula (2). From the figure it is seen that the maximum velocity corresponds to the center of the vortex ring (to the coordinate origin, point 0). In the remaining directions from the center, the velocity decreases asymptotically till zero.

The streamlines of the vortex moving in a rectilinear flow with the velocity $V_1 = nV_0$ are depicted in Fig. 2, where $n = 0.09$. The trajectories of fluid particles were calculated from formula (6). The streamline corresponding to the value $\Psi = 0$ splits into a straight line, which is the axis of symmetry of the torus (the $0z$ axis), and into a circle of radius

$$R = \frac{a}{k}\sqrt{1 - k^2}\,,$$

obtained from formula (6), where $k^4 = n$. Thus, in addition to circulation about the vortex axis (point A), the fluid in the interior of this circle moves with the velocity V_1 in the direction opposite to the $0z$ axis. The axis of the vortex, i.e., the circle where $V = 0$, "has contracted" around the symmetry axis. In Fig. 1, $0A = 1$, and in Fig. 2 it is 0.787.

As the velocity of the torus V_1 increases, the radius R of the line $\Psi = 0$ decreases, and at $k = 1$ (i.e., at $V_1 = V_0$) it disappears ($R = 0$). This means that one will not observe a toroidal vortex as such. Thus, the value of the velocity of the torus in the fluid lies in the range $0 < V_1 < V_0$, i.e., it depends on the initial circulation determined by V_0.

We calculate the circulation along the boundary between two flow regions from the formula

$$\Gamma = \oint_{BCD} U_\theta dl + \oint_{DB} U_z dz\,,$$

where U_θ is the velocity of the fluid at the BCD boundary in polar coordinates with the origin at the point 0. The result of calculations is

$$\Gamma = V_0 R k^2 (2k^2 - 4k^4 + 1) + V_0 a \arctan\frac{\sqrt{1 - k^2}}{k}\,.$$

In the case of a stationary vortex ring, i.e., at $k = 0$, we obtain an expression

$$\Gamma = V_0 \pi a/2\,,$$

which corresponds to (5).

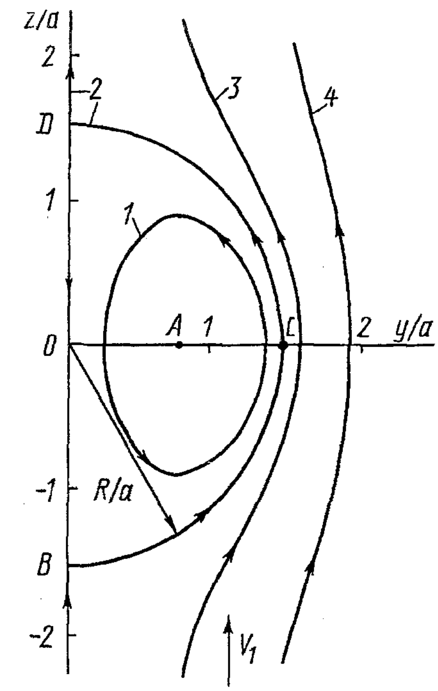

Fig. 2. Distribution of streamlines $\Psi / V_0 a^2$ in flow around a vortex ring: 1) $\Psi / V_0 a^2 = 0.025$; 2) 0; 3) 0.025; 4) 0.1.

Proceeding from the Bernoulli equation, we obtain a formula for the streamline to calculate the pressure

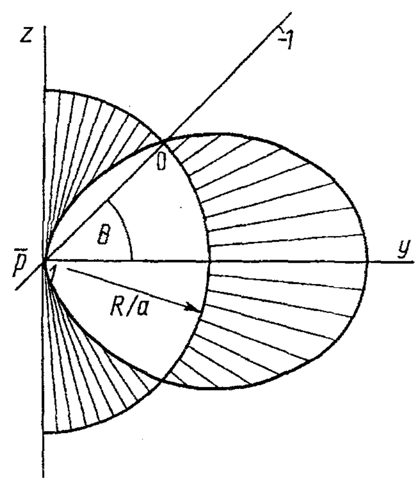

Fig. 3. Change in the pressure difference $\bar{p}$ along the boundary as a function of θ, rad.

difference along the line $\Psi = 0$:

$$\bar{p} = 2\frac{p - p_0}{\rho V_1^2} = 1 - \left(\frac{U}{V_1}\right)^2 .$$

Just as in the case of calculation of the circulation Γ, it is convenient to represent the velocity U in polar coordinates (r, θ) in the $y0z$ plane. Then, the velocity components U_r and U_θ at $r = R$ will be written as

$$U_r = 0 , \quad U_\theta = 2k^4 V_0 (1 - k^2) \cos\theta .$$

The results of computations of $\bar{p}$ at the boundary between two flow regions are presented in Fig. 3. It is seen from the figure that the vortical region contracts along the symmetry axis and expands at the edges. Such a distribution of pressures explains the expansion of the vortex ring in a transverse direction during the motion of the torus which is visually observed [18].

NOTATION

σ, τ, φ, toroidal coordinates; $\overline{V}$ (V_σ; V_τ; V_φ), velocity of a fluid particle and its projections in toroidal coordinates; $g_{\sigma\sigma}$, $g_{\tau\tau}$, $g_{\varphi\varphi}$, metric tensor components; $\sqrt{g} = \sqrt{g_{\sigma\sigma}g_{\tau\tau}g_{\varphi\varphi}}$, the Jacobian of transition to curvilinear coordinates; V_0, velocity at the center of a vortex ring on its symmetry axis; x, y, z, Cartesian coordinates; z, y, φ, cylindrical coordinates; a, distance from the axis of a torus ($V = 0$) to its axis of symmetry (0z); α, angle between the 0y axis and the line that connects a fluid particle on the streamline τ = const, which represents a circle [16], with the center of this circle; U_z, U_y, velocities in the cylindrical system of coordinates; Ψ_0, stream function of a stationary vortex ring; Γ, velocity circulation; $U_\infty = V_1$, velocity of a rectilinear flow at infinity; Ψ_1, stream function of a rectilinear flow; $\Psi = \Psi_0 + \Psi_1$, superposition of two flows; $n = k^4 = V_1/V_0$, velocity ratio coefficient; R, radius of a vortical region; U_θ, velocity of fluid particles at the boundary in polar coordinates (r, θ) with the center at the coordinate origin (point 0); ρ, fluid density; p_0, p, pressure at infinity and at a certain point of flow; $\overline{p}$, pressure difference.

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